Introduction to FEM

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Contents of this section

- Weak form of differential equations
- Galerkin finite element discretisation
- Boundary conditions
- All done through the example of the Poisson equation

Model equation

Navier-Stokes equations:

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u} = 0$$

Take divergence of momentum equation:

$$\frac{\partial}{\partial t} \nabla \cdot \boldsymbol{u} + \nabla \cdot ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) = -\nabla^2 \boldsymbol{p} + \nu \nabla^2 \nabla \cdot \boldsymbol{u}$$

Given a velocity field, we solve for the pressure.

Model equation

We will write equation in a more general form:

$$\nabla^2 \boldsymbol{u} = f(\boldsymbol{x})$$

...and we'll consider the one-dimensional version:

$$u_{xx} = f(x)$$

We'll call these forms of the equations the "strong form"

Strong form

The “strong” way to check if u is a solution: check if the equation is true at each point in space

To make a finite difference method, we approximate the derivatives, and check the approximate equation is true at each gridpoint

Weak form

The “weak” way is to multiply the equation by a “test function” and then integrate over the domain:

$$\int_\Omega \boldsymbol{\phi} \nabla^2 \boldsymbol{u} \, \mathrm{d}\, V = \int_\Omega \boldsymbol{\phi} f \, \mathrm{d}\, V$$

Integrate by parts:

$$-\int_\Omega \nabla \phi \cdot \nabla \boldsymbol{u} \, \mathrm{d}\, V + \int_{\partial\Omega} \phi \boldsymbol{n} \cdot \nabla \boldsymbol{u} \, \mathrm{d}\, S = \int_\Omega \boldsymbol{\phi} f \, \mathrm{d}\, V$$

Check this is true for all functions where integral can be evaluated

Strong and weak form

- If u satisfies the strong form, then it satisfies the weak form
- If u satisfies the weak form for all test functions, and u is **suitably smooth**, then u satisfies the strong form
- To make an FEM, we keep the weak form, but **restrict** the types of functions that we allow for u and test functions

Descriptive Formulation

We consider the one-dimensional Poisson equation

$$L(u) \equiv \frac{\partial^2 u}{\partial x^2} + f = 0.$$

Boundary conditions

For this problem to be well posed and therefore have a unique solution we need to specify boundary conditions.

$$\Omega = \{x \mid 0 < x < 1\}$$

$$u(0) = g_D, \quad \frac{\partial u}{\partial x}(1) = g_N,$$

In the Galerkin formulation, Dirichlet boundary conditions have to be specified explicitly whereas Neumann conditions are dealt with implicitly as part of the formulation

If the boundary conditions stated above are applied to the Poisson we have a two-point boundary value problem and is said to be in the strong (classical) form

Weak form

Multiply equation by a weight/test function (which vanishes on all Dirichlet boundaries) and integrate:

$$(v, L(u)) = \int_0^1 v \left(\frac{\partial^2 u}{\partial x^2} + f \right) dx = 0$$

Equivalent to setting the weighted residual to zero.

Integrating by parts gives

$$\int_0^1 \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} dx = \int_0^1 v f \; dx + \left[v \frac{\partial u}{\partial x} \right]_0^1 .$$

Natural boundary conditions

- This is a common approach in finite elements, it reduces the order of the second derivative and makes the matrix system symmetric.
- As the test functions are defined to be zero on Dirichlet boundaries we know that $v(0) = 0$.

Natural boundary conditions

if we apply the Neumann boundary condition

$$\partial u(1)/\partial x = g_N$$

to the last term, we get

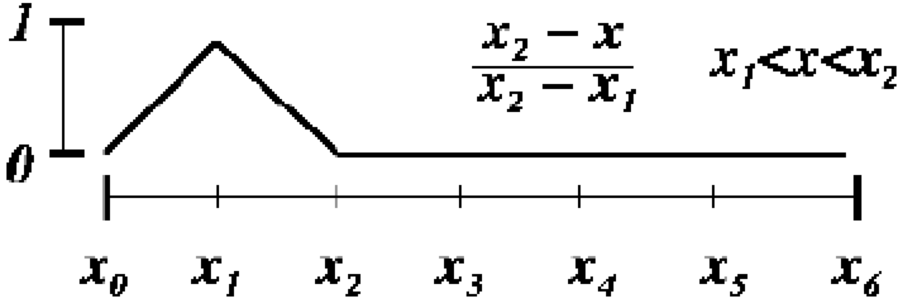
$$\int_0^1 \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} dx = \int_0^1 vf \; dx + v(1) g_N.$$

Neumann boundary conditions are naturally included in the formulation

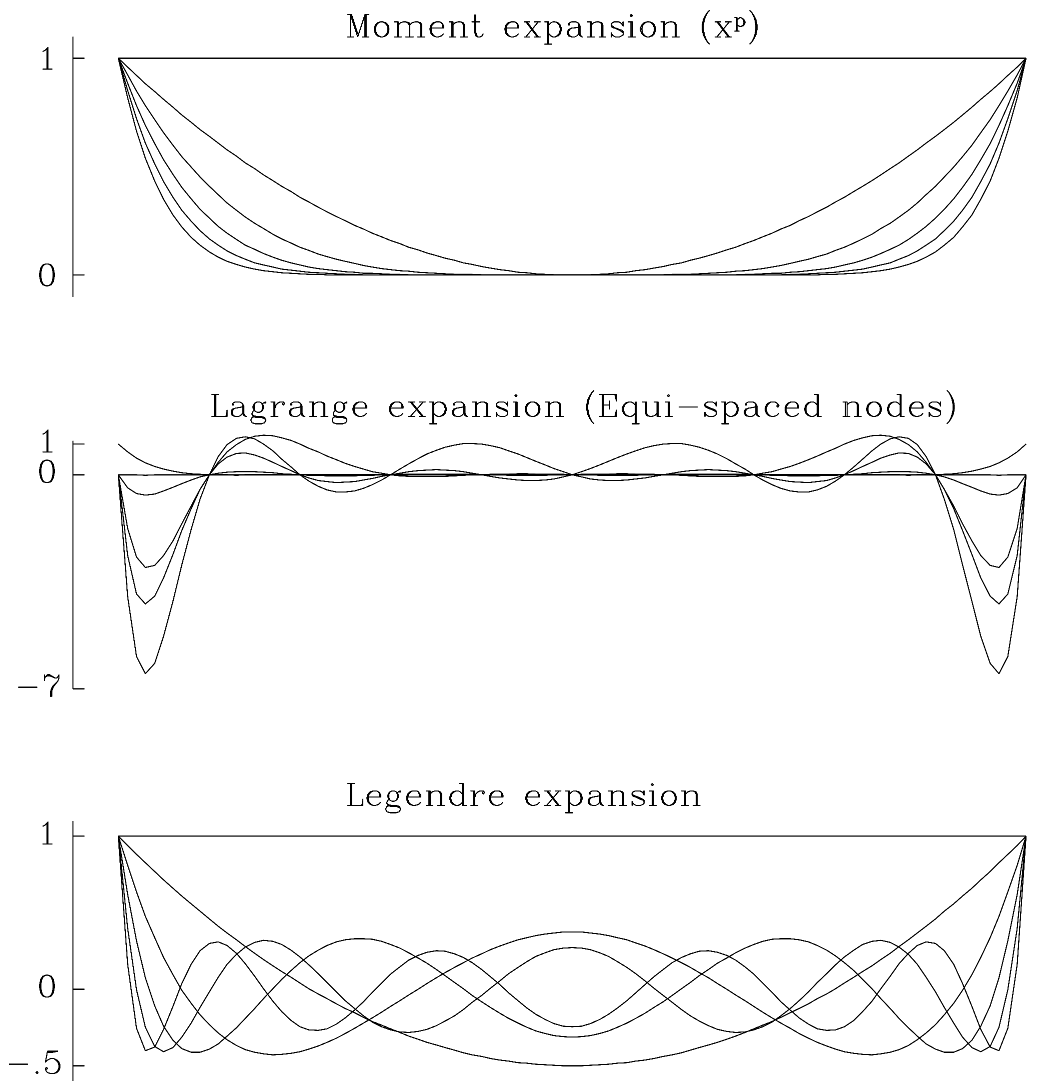
Finite element functions

We represent our solution as $u(x) = \sum_{i=1}^{N} \hat{u}_i N_i(x)$

$$N_1(x) = \begin{cases} \dfrac{x - x_0}{x_1 - x_0} & x_0<x<x_1 \\[2ex] \dfrac{x_2 - x}{x_2 - x_1} & x_1<x<x_2 \end{cases}$$

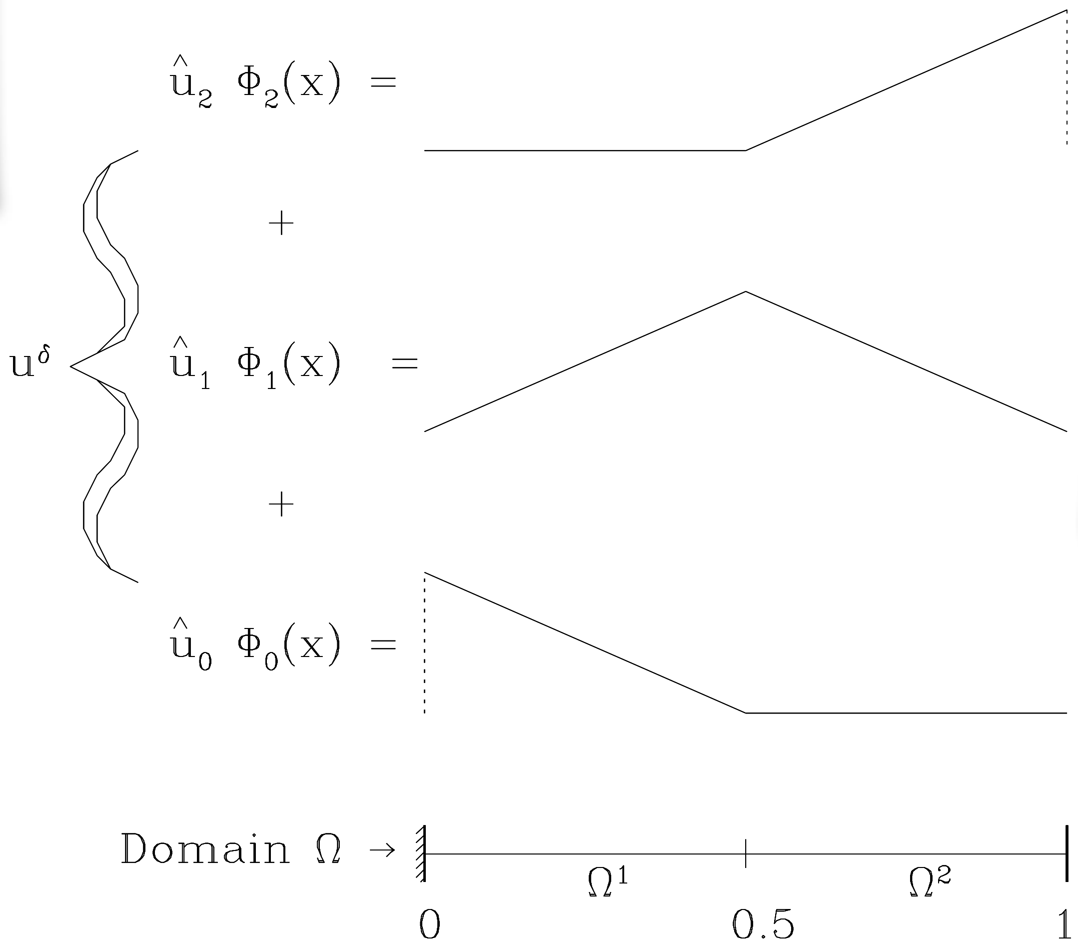


Example modes



Shape functions

reproduce diagram



h-type approximation
h-parameter represents size of domain
convergence achieved by subdividing domain into smaller and smaller subdomains so that
h -> 0

Galerkin approximation

The Galerkin approximation of the Poisson equation is the solution to the weak form when the exact solution is approximated by our finite element expansion of *u*

The weight or test function is also replaced by a finite expansion, and we get

$$\int_0^1 \frac{\partial v^\delta}{\partial x}\frac{\partial u^\delta}{\partial x}dx = \int_0^1 v^\delta f \; dx + v^\delta(1)g_N$$

Functions used in *u* are referred to as the trial functions whereas functions used in *v* are referred to as the test functions

Dirichlet conditions

- The approximate solution contains some functions which are zero on Dirichlet boundaries and some which are not.
- Therefore construct solution from a known function which satisfies the Dirichlet boundary conditions and a homogeneous solution which is zero on the Dirichlet boundaries

$$u^{\mathcal{H}}(\partial\Omega_{\mathcal{D}}) = 0, \quad u^{\mathcal{D}}(\partial\Omega_{\mathcal{D}}) = g_D.$$

Dirichlet conditions

- The same set of basis functions are now used to represent the homogeneous solution and the test functions
- Substitution gives

$$\int_0^1 \frac{\partial v^\delta}{\partial x}\frac{\partial u}{\partial x}^{\mathcal{H}} dx = \int_0^1 v^\delta f \; dx + v^\delta(1) g_N - \int_0^1 \frac{\partial v^\delta}{\partial x}\frac{\partial u}{\partial x}^{\mathcal{D}} dx.$$

Two-Domain Linear Finite Element Example

In this section we solve the one-dimensional Poisson equation

$$L(u) \equiv \frac{\partial^2 u}{\partial x^2} + f = 0,$$

emphasise
boundary
condition

where *f* is a known function and boundary conditions are

$$u(0) = g_D = 1, \quad \frac{\partial u}{\partial x}(1) = g_N = 1.$$

Weak formulation

We start by considering the weak form

$$\int_0^1 \frac{\partial v^\delta}{\partial x}\frac{\partial u}{\partial x}^{\mathcal{H}} dx = \int_0^1 v^\delta f \; dx + v^\delta(1) g_N - \int_0^1 \frac{\partial v^\delta}{\partial x}\frac{\partial u}{\partial x}^{\mathcal{D}} dx.$$

Link to matrix equations

Finite Element basis

$$u^\delta = \sum_{i=0}^{2} \hat{u}_i N_i(x),$$

emphasise
two
domains

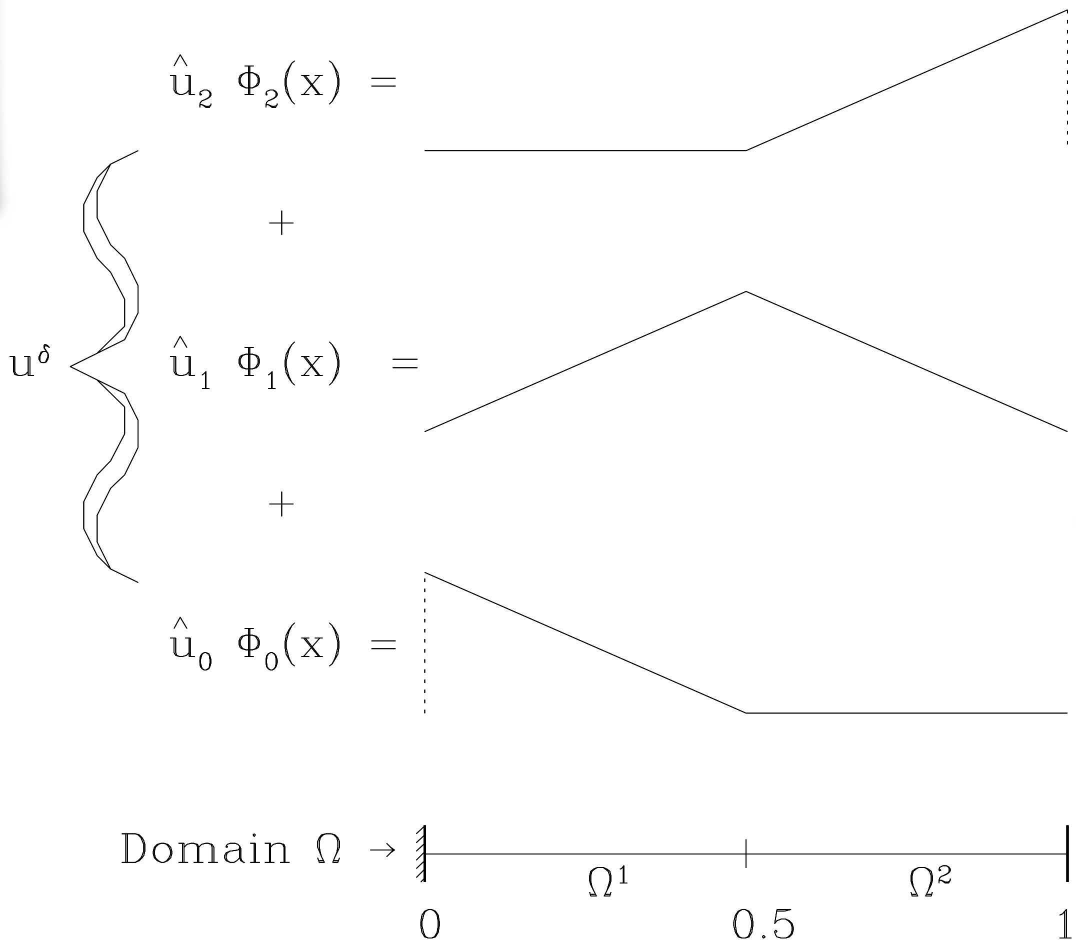
$$N_0(x) = \begin{cases} 1-2x & 0 \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} \le x \le 1 \end{cases}$$

$$N_1(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \le x \le 1 \end{cases}$$

$$N_2(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} \\ 2x-1 & \frac{1}{2} \le x \le 1 \end{cases}$$

Shape functions

reproduce diagram



h-type approximation
h-parameter represents size of domain
convergence achieved by subdividing domain into smaller and smaller subdomains so that h -> 0

Global to Local

- We have simplified the problem by considering N as a global expansion.

- However the great power of the finite element method is its geometric flexibility arising from decomposing the global expansions into local expansions.

Dirichlet Condition

The only way to satisfy the Dirichlet condition at $x = 0$ is to set $\hat{u}_0 = g_D$

To do this, we decompose u^δ into $u^\delta = u^{\mathcal{H}} + u^{\mathcal{D}}$

$$\begin{aligned} u^{\mathcal{H}} &= \hat{u}_1 N_1(x) + \hat{u}_2 N_2(x) \\ u^{\mathcal{D}} &= g_D N_0(x), \end{aligned}$$

have reduced degrees of freedom by 1

look for a solution with u=0 at x=0 and add

Galerkin discretisation

In the Galerkin approach the same expansion bases are used to define the test function

never have to determine
v^1 v^2

$$v^\delta(x) = \hat{v}_1 N_1(x) + \hat{v}_2 N_2(x).$$

Right-hand side

f is known explicitly and therefore it is theoretically possible to evaluate this term exactly. In practice the function is usually represented using the same expansion as *u*

$$f^\delta(x) = \sum_{i=0}^{2} \hat{f}_i N_i(x) = \hat{f}_0 N_0(x) + \hat{f}_1 N_1(x) + \hat{f}_2 N_2(x).$$

for our problem

$$\hat{f}_0 = f(0), \hat{f}_1 = f(0.5), \hat{f}_2 = f(1)$$

Calculating integrals

$$
\begin{aligned}
\int_0^1 \frac{\partial v^\delta}{\partial x}\frac{\partial u}{\partial x}^{\mathcal{H}} dx &= \int_0^{\frac{1}{2}} (2\hat{v}_1)(2\hat{u}_1)dx + \int_{\frac{1}{2}}^1 (-2\hat{v}_1 + 2\hat{v}_2)(-2\hat{u}_1 + 2\hat{u}_2)dx \\
&= \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} \\
\int_0^1 v^\delta f \; dx &= \int_0^{\frac{1}{2}} (\hat{v}_1 2x)(\hat{f}_0(1-2x) + \hat{f}_1(2x))dx \\
&+ \int_{\frac{1}{2}}^1 (\hat{v}_1 2(1-x) + \hat{v}_2(2x-1))(\hat{f}_1 2(1-x) + \hat{f}_2(2x-1))dx \\
&= \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \end{bmatrix} \begin{bmatrix} \frac{1}{12}\hat{f}_0 + \frac{1}{3}\hat{f}_1 + \frac{1}{12}\hat{f}_2 \\ \frac{1}{12}\hat{f}_1 + \frac{1}{6}\hat{f}_2 \end{bmatrix} \\
v^\delta(1)g_N &= (\hat{v}_1 N_1(1) + \hat{v}_2 N_2(1))g_N = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} g_N \\
\int_0^1 \frac{\partial v^\delta}{\partial x}\frac{\partial u}{\partial x}^{\mathcal{D}} dx &= \int_0^{\frac{1}{2}} (2\hat{v}_1)(-2g_D)dx = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \end{bmatrix} \begin{bmatrix} -2g_D \\ 0 \end{bmatrix}.
\end{aligned}
$$

link to weak form

Matrix equations

$$\begin{bmatrix} \hat{v}_1 & \hat{v}_2 \end{bmatrix} \left\{ \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} - \begin{bmatrix} \frac{1}{12}\hat{f}_0 + \frac{1}{3}\hat{f}_1 + \frac{1}{12}\hat{f}_2 \\ \\ \frac{1}{12}\hat{f}_1 + \frac{1}{6}\hat{f}_2 \end{bmatrix} \right.$$

$$\left. - \begin{bmatrix} 0 \\ g_N \end{bmatrix} + \begin{bmatrix} -2g_D \\ 0 \end{bmatrix} \right\} = 0.$$

This equation has to be true for all test functions, so we get the matrix equation in the curly brackets

Boundary conditions

Recalling that $g_D = 1$ and $g_N = 1$ we get

$$\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} = \begin{bmatrix} 2 + \frac{1}{12}\hat{f}_0 + \frac{1}{3}\hat{f}_1 + \frac{1}{12}\hat{f}_2 \\ \\ 1 + \frac{1}{12}\hat{f}_1 + \frac{1}{6}\hat{f}_2 \end{bmatrix}$$

which has a solution

$$\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \frac{1}{24}\hat{f}_0 + \frac{5}{24}\hat{f}_1 + \frac{1}{8}\hat{f}_2 \\ \\ 2 + \frac{1}{24}\hat{f}_0 + \frac{1}{4}\hat{f}_1 + \frac{5}{24}\hat{f}_2 \end{bmatrix}.$$

Solution

The finite element approximation

$$u^\delta(x) = g_D N_0(x) + \hat{u}_1 N_1(x) + \hat{u}_2 N_2(x)$$

is

$$u^\delta = \begin{cases} 1 + x + \dfrac{x}{12}\hat{f}_0 + \dfrac{5x}{12}\hat{f}_1 + \dfrac{x}{4}\hat{f}_2 & 0 \le x \le \frac{1}{2} \\ 1 + x + \dfrac{1}{24}\hat{f}_0 + \dfrac{2+x}{12}\hat{f}_1 + \dfrac{1+4x}{24}\hat{f}_2 & \frac{1}{2} \le x \le 1 \end{cases}$$

Mathematical Formulation

We consider the one-dimensional Helmholtz equation

$$L(u) = \frac{\partial^2 u}{\partial x^2} - \lambda u + f = 0,$$

where λ is a real positive constant

To be solved in the domain $0 < x < l$

The equation comes with boundary conditions

$$u(0) = g_D, \quad \frac{\partial u}{\partial x}(l) = g_N.$$

Integral form

Multiplying by an arbitrary test function, the properties of which are to be defined, and integrating over the domain we obtain

$$\int_0^l v\frac{\partial^2 u}{\partial x^2} - \int_0^l \lambda v u \; dx + \int_0^l v f \; dx = 0.$$

If the functions are smooth enough, we can integrate by parts

$$\int_0^l \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \int_0^l \lambda v u \; dx = \int_0^l v f \; dx + \left[v\frac{\partial u}{\partial x} \right]_0^l .$$

Bilinear form notation

$$
\begin{aligned}
a(v,u) &= \int_0^l \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \lambda v u\right) dx \\
f(v) &= \int_0^l v f \, dx + \left[v \frac{\partial u}{\partial x}\right]_0^l
\end{aligned}
$$

We write the equation

$$a(v,u) = f(v)$$

Strain energy

In structural mechanics, $a(v,u)$ is known as the strain energy

the space of all functions which have a finite strain is called the energy space denoted by

$$E(\Omega) = \{u \mid a(u,u) < \infty\}$$

Associated with the energy space is the energy norm

$$||u||_E = \sqrt{a(u,u)}.$$

Functions that belong to the energy space are called H^1 functions and satisfy the condition that the integral of the square of the function and its derivative are bounded.

Trial and test functions

For our problem the trial space is defined by

physical condition
E bounded

boundary condition

$$\mathcal{X} = \{u \mid u \in H^1, u(0) = g_D\}.$$

The space of all test functions which are homogeneous on Dirichlet boundaries is

$$\mathcal{V} = \{v \mid v \in H^1, v(0) = 0\}.$$

Weak formulation

We can now define the generalized or weak formulation of our equation

Find $u \in \mathcal{X}$ such that

$$a(v, u) = f(v), \qquad \forall\, v \in \mathcal{V}.$$

Approximate weak form

We select subspaces $\mathcal{X}^\delta \subset \mathcal{X}, \quad \mathcal{V}^\delta \subset \mathcal{V}$
with a finite number of degrees of freedom

The approximate form of the weak formulation is

Find $u^\delta \in \mathcal{X}^\delta$ such that

$$a(v^\delta, u^\delta) = f(v^\delta) \qquad \forall\ v^\delta \in \mathcal{V}^\delta.$$

Galerkin formulation

D is known with dirichlet
H is unknown with
dirichlet zero
boundary conditions go
to right hand side
u^H identical to v
functions with zero bcs

Find

$$u^\delta = u^{\mathcal{D}} + u^{\mathcal{H}}, \text{ where } u^{\mathcal{H}} \in \mathcal{V}^\delta,$$

such that

$$a(v^\delta, u^{\mathcal{H}}) = f(v^\delta) - a(v^\delta, u^{\mathcal{D}}) \text{ for all } v^\delta \in \mathcal{V}^\delta$$

Bilinear forms

We have a symmetric bilinear form

$$
\begin{aligned}
a(v,u) &= a(u,v) \\
a(c_1 v + c_2 w, u) &= c_1 a(v,u) + c_2 a(w,u),
\end{aligned}
$$

The equation is *bounded* if

$$|a(v,u)| \leq C_1 ||v||_1 ||u||_1, \quad C_1 < \infty$$

The equation is *elliptic* if

$$a(u,u) \geq C_2 ||v||_1^2, \quad C_2 > 0$$

Uniqueness of Galerkin approximation

Assume that there are two distinct solutions

$$a(v^\delta, u_1) = f(v^\delta), \qquad \text{for all } v^\delta \in \mathcal{V}^\delta$$
$$a(v^\delta, u_2) = f(v^\delta), \qquad \text{for all } v^\delta \in \mathcal{V}^\delta.$$

Subtracting gives

$$a(v^\delta, u_1) - a(v^\delta, u_2) = a(v^\delta, u_1 - u_2) = 0$$

Now choose $v^\delta = u_1 - u_2$

$$0 = a(u_1 - u_2, u_1 - u_2) \| u_1 - u$$

Contradiction!

why do we care?

because if non-unique then we can't invert the matrix and get an answer!

Orthogonality of error

$$a(v^\delta, \varepsilon) = 0, \quad \forall\, v^\delta \in \mathcal{V}^\delta, \quad \varepsilon = u - u^\delta$$

To prove this, note that the approximate trial space is contained in the full trial space

$$a(v^\delta, u) = f(v^\delta), \qquad\qquad \forall\, v^\delta \in \mathcal{V}^\delta.$$

subtracting the approximate equation

$$a(v^\delta, u^\delta) = f(v^\delta), \qquad\qquad \forall\, v^\delta \in \mathcal{V}^\delta.$$

Gives the result.

Energy minimiser

solution minimises error in energy norm

$$||u - u^\delta||_E = \min_{w^\delta \in \mathcal{X}^\delta} ||u - w^\delta||_E.$$

why do we care? this shows minimum residual leads to minimum error

To prove this, note that for any $w^\delta \in \mathcal{X}^\delta$ we have

$$||u - w^\delta||_E^2 = ||u - u^\delta + u^\delta - w^\delta||_E^2 = ||\varepsilon + v^\delta||_E^2$$

where $v^\delta = w^\delta - u^\delta \in \mathcal{V}^\delta$

$$||u - w^\delta||_E^2 = a(\varepsilon + v^\delta, \varepsilon + v^\delta) = a(\varepsilon, \varepsilon) + 2a(v^\delta, \varepsilon) + a(v^\delta, v^\delta).$$

a(v^d,e) vanishes a(e,e) is energy norm

So the error is minimised over the trial space in the energy norm

Equivalence of bases

Uniqueness of the Galerkin approximation means that any two linearly independent expansions with the same trial space have the same approximate solution u.

Two solutions to Galerkin approximation

$$\begin{aligned} u_1^\delta(x) &= \textstyle\sum_i^P \alpha_i \psi_i(x) \\ u_2^\delta(x) &= \textstyle\sum_i^P \beta_i h_i(x) \end{aligned}$$

$$u_1^\delta(x) = u_2^\delta(x) \quad \Rightarrow \quad \sum_{i=0}^{P} \alpha_i \psi_i(x) = \sum_{i=0}^{P} \beta_i h_i(x).$$

Equivalence of bases

- Important implication is that any error estimates are independent of the type of the polynomial expansion and only depend on the polynomial space.
- Different choices of polynomial expansion bases can have an important effect on the numerical conditioning of matrix systems.

how easy it is to invert matrix accurately